ON THE EXISTENCE OF DISTORTION MAPS ON ORDINARY ELLIPTIC CURVES

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1. Introduction

An important problem in cryptography is the so called Decision Diffie-Hellman problem (henceforth abbreviated DDH). The problem is to distinguish triples of the form (g^a, g^b, g^{ab}) from arbitrary triples from a cyclic group $G = \langle g \rangle$. It turns out that for (cyclic subgroups of) the group of m-torsion points on an elliptic curve over a finite field, the DDH problem admits an efficient solution if there exists a suitable endomorphism called a distortion map (which can be efficiently computed) on the elliptic curve.

Suppose m is relatively prime to the characteristic of a finite field $\mathbb{F}_q$, then the group of m-torsion points on an elliptic curve $E/\mathbb{F}_q$, denoted $E[m]$, is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$. Fix an elliptic curve $E/\mathbb{F}_q$ and a prime ℓ that is not the characteristic of $\mathbb{F}_q$. Let P and Q generate the group $E[\ell]$. A *distortion map* on E is an endomorphism ϕ of E such that $\phi(P) \notin \langle P \rangle$. A distortion map can be used to solve the DDH problem on the group $\langle P \rangle$ as follows: Given a triple R, S, T of points belonging to the group generated by P, we check whether $\mathbf{e}_\ell(R, \phi(S)) = \mathbf{e}_\ell(P, \phi(T))$, where $\mathbf{e}_\ell$ is the Weil pairing on the ℓ-torsion points. It follows from well known properties of the Weil pairing that this check succeeds if and only if $R = aP$, $S = bP$ and $T = abP$. Under the assumptions that P and Q are both defined over $\mathbb{F}_{q^k}$, where k is not large (say, bounded by a fixed polynomial in $\log(q)$), and that ϕ can be computed in polynomial time, the DDH problem can be solved in polynomial time using this idea. If P and Q are not eigenvectors for the Frobenius map, then in many cases one can use the trace map as a distortion map (see [GR04]). For this reason, we will concentrate only on the subgroups that are Frobenius eigenspaces.

It is known that distortion maps exist on supersingular elliptic curves ([Ver01, GR04]), and that distortion maps that do not commute with the Frobenius do not exist on ordinary elliptic curves (see [Ver01] or [Ver04] Theorem 6). The latter implies that distortion maps *do not* exist for ordinary elliptic curves with embedding degree > 1. The embedding degree, (say) k, is the order of q in the group $(\mathbb{Z}/\ell\mathbb{Z})^*$. A theorem of Balasubramanian and Koblitz ([BK98] Theorem 1) says that if $E(\mathbb{F}_q)$ contains an ℓ-torsion point and $k > 1$, then $E[\ell] \subseteq \mathbb{F}_{q^k}$. Thus, the only remaining cases where the existence of Distortion maps is not known are the cases when the embedding degree k is 1. If the embedding degree is 1 and $E(\mathbb{F}_q)$ contains an ℓ-torsion point, then there are two possibilities: either $E[\ell](\mathbb{F}_q)$ is cyclic or $E[\ell] \subseteq E(\mathbb{F}_q)$. In the former situation there are no distortion maps (by [Ver04] Theorem 6). However, the Tate pairing can be used to solve DDH efficiently in this case (see the comments in [GR04] following Remark 2.2). Thus, the only case in which the question of the existence of a distortion map remains open is when $E[\ell] \subseteq E(\mathbb{F}_q)$. In this article we characterize the existence of distortion maps for this case.

2. The Proof

Let k be a finite field, $\mathbb{F}_q \supseteq k$ and E/k be an ordinary elliptic curve. Suppose ℓ is a prime such that $E[\ell] \subseteq \mathbb{F}_q$ but no point of exact order ℓ is defined over a smaller field.

To study the existence of distortion maps, we study the reduction of the ring $\mathrm{End}(E)$ modulo ℓ. Our principal tool is the following observation: If $\alpha \in \mathrm{End}(E)$ has field polynomial $f(x) \in \mathbb{Z}[x]$, then $f \bmod \ell$ is the characteristic equation of the action of α on $E[\ell]$.

Let π be the q-th power Frobenius endomorphism on E and let $\phi^2 - t\phi + q = 0$ be its characteristic equation. We know that $t \equiv 2 \mod \ell$ and $q \equiv 1 \mod \ell$ as the full ℓ-torsion is defined over $\mathbb{F}_q$.

Let $\mathcal{O} = \mathrm{End}(E)$, $K = \mathcal{O} \otimes \mathbb{Q}$ and $\mathcal{O}_K$ the maximal order in K. We have the inclusions $\mathbb{Z}[\pi] \subseteq \mathcal{O} \subseteq \mathcal{O}_K$. Since $t^2 - 4q = 0 \mod \ell$ we have that ℓ divides the product $[\mathcal{O} : \mathbb{Z}[\pi]][\mathcal{O}_K : \mathcal{O}]\mathrm{Disc}(K)$. The existence of distortion maps splits into cases depending on whether $\ell | [\mathcal{O}_K : \mathcal{O}]$ or $\ell | \mathrm{Disc}(K)$. Indeed, if $\ell | [\mathcal{O}_K : \mathcal{O}]$ there are no distortion maps, since the reduction modulo ℓ of every endomorphism is just multiplication by scalar.

In the following we assume that $\ell \not| [\mathcal{O}_K : \mathcal{O}]$ so that the conductor of $\mathcal{O}$ is prime to ℓ. Under this assumption we have that the residue class rings

$$\mathcal{O}_K/(\ell) \cong \mathcal{O}/(\ell).$$

Suppose that $\ell \not| \mathrm{Disc}(K)$ and that ℓ is *inert* in $\mathcal{O}_K$, then $\mathcal{O}/(\ell) \cong \mathbb{F}_{\ell^2}$. Let $\alpha \in \mathcal{O}$ be an endomorphism such that $\alpha \mod (\ell)$ does not lie in $\mathbb{F}_\ell$. Then the action of α on $E[\ell]$ is irreducible since its characteristic equation is irreducible over $\mathbb{F}_\ell$. Now α gives us a distortion map on $E[\ell]$ since no subgroup of order ℓ of $E[\ell]$ is stabilized by α.

Now if $\ell \not| \mathrm{Disc}(K)$ and ℓ is *split* in $\mathcal{O}_K$, then $\mathcal{O}/(\ell) \cong \mathbb{F}_\ell[X]/(X-a)(X-b) \cong (\mathbb{Z}/\ell\mathbb{Z})^2$ (where $a \neq b$). The action of any $\alpha \in O_K$, that corresponds to the image of X in $\mathbb{F}_\ell[X]/(X-a)(X-b)$ under the isomorphism, is conjugate to $\begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}$. Thus, distortion maps exist for all but two of the subgroups of $E[\ell]$.

Suppose that $\ell | \mathrm{Disc}(K)$ so that ℓ is *ramified* in $\mathcal{O}_K$, then $\mathcal{O}/(\ell) \cong \mathbb{F}_\ell[X]/(X-a)^2$. Consider the map $\alpha \in \mathcal{O}$ that corresponds to the image of X in the ring $\mathbb{F}_\ell[X]/(X-a)^2$. The action of α on $E[\ell]$ is conjugate to $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Note that $\beta \neq 0$, for if $\beta = 0$ then $\mathcal{O}/(\ell) \cong \mathbb{Z}/\ell\mathbb{Z}$, but we know that $\mathcal{O}$ is rank 2 over $\mathbb{Z}/\ell\mathbb{Z}$ since ℓ is ramified in $\mathcal{O}_K$ and does not divide the conductor of $\mathcal{O}$. Thus, distortion maps exist for all but one subgroup of $E[\ell]$.

In summary, we have:

Theorem 2.1. *Let k be a finite field, $\mathbb{F}_q \supseteq k$ and E/k be an ordinary elliptic curve whose endomorphism ring is $\mathcal{O}$, an order in an imaginary quadratic field $\mathcal{O}$. Suppose ℓ is a prime such that $E[\ell] \subseteq \mathbb{F}_q$ but no point of exact order ℓ is defined over a smaller field.*

(1) *If $\ell \mid [\mathcal{O}_K : \mathcal{O}]$ there are no distortion maps.*
(2) *If $\ell \not| [\mathcal{O}_K : \mathcal{O}]\mathrm{Disc}(K)$ and*
 (a) *ℓ is inert in $\mathcal{O}_K$, then there are distortion maps for every (order ℓ) subgroup of $E[\ell]$;*
 (b) *ℓ is split in $\mathcal{O}_K$, then all but two subgroups of $E[\ell]$ have distortion maps.*
(3) *If $\ell \not| [\mathcal{O}_K : \mathcal{O}]$ and $\ell \mid \mathrm{Disc}(K)$ so that ℓ is ramified in $\mathcal{O}_K$, then all (except one) subgroups of $E[\ell]$ have distortion maps.*

3. Examples

In this section, we give examples to illustrate that all the cases in Theorem 2.1 do occur.

Example 3.1. Consider the elliptic curve $E : y^2 = x^3 + x$ over $\mathbb{Q}$. E has complex multiplication by $\mathbb{Z}[\imath]$ and has good reduction at all odd primes. Let p be a prime such that $p \equiv 1 \mod 4$, $\tilde{E}$ be the reduction of E modulo p, and let $\imath^2 = -1 \mod p$. Then $\tilde{E}[2] \subseteq \tilde{E}(\mathbb{F}_p)$ and $\tilde{E}[2]$ is $\{0_{\tilde{E}}, (0,0), (\imath, 0), (-\imath, 0)\}$ where $0_{\tilde{E}}$ is the identity element. The map $[\imath]$ is an endomorphism that sends $(x, y) \mapsto (-x, \imath y)$. It is easy to see that the map $[\imath]$ preserves the subgroup $\langle(0,0)\rangle$ and interchanges the remaining two subgroups, of order 2, of $\tilde{E}[2]$. Note, that Deuring's reduction theorem tells us that $\mathrm{End}(\tilde{E}) \cong \mathbb{Z}[i]$. Furthermore, in this case the subring $\mathbb{Z}[\pi]$ generated by the Frobenius is usually a smaller ring. Indeed, if t is the trace of Frobenius and $t^2 - 4p = -4b^2$, then the conductor of the order $\mathbb{Z}[\pi]$ is b. Now b is at least 2, since $t \equiv 2 \mod 4$, so $(t/2)$ is odd and we must have $p = (t/2)^2 + b^2$. Thus, case (3) of Theorem 2.1 applies and matches with what we observe for the 2-torsion.

Example 3.2. (Suggested by anonymous reviewer). Let E be the curve over $\mathbb{F}_{701}$ given by the equation $y^2 = x^3 - 35x + 98$. Then $\mathrm{End}(E) = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ which is the maximal order in $\mathbb{Q}(\sqrt{-7})$. The order $\mathbb{Z}[\pi]$

has conductor 10 in $\mathrm{End}(E)$. The 5-torsion is $\mathbb{F}_{701}$ rational, and moreover, 5 is inert in $\mathrm{End}(E)$. Theorem 2.1 (2a) shows that every subgroup of $E[5]$ admits a distortion map. Indeed, the map corresponding to multiplication by $\alpha = \frac{1+\sqrt{-7}}{2}$ is given by ([Sil94] Chapter II, Proposition 2.3.1 (iii))

$$[\alpha](x, y) = \left(\alpha^{-2}\left(x - \frac{7(1-\alpha)^4}{x+\alpha^2-2}\right), \alpha^{-3}y\left(1 + \frac{7(1-\alpha)^4}{(x+\alpha^2-2)^2}\right)\right).$$

Let us check this for the group generated by the 5-torsion point P (with affine coordinates) $P = (224, 31)$. Since $\alpha = 386 \in \mathbb{F}_{701}$, this tells us that $[\alpha](P) = (173, 194)$. One checks that the Weil pairing $\mathbf{e}_5(P, [\alpha](P)) = 464 \neq 1$. Thus, $[\alpha]$ works as a distortion map for the group generated by P.

Now the 5-torsion of E is generated by P and the point $Q = (573, 450)$. A similar computation shows that $[\alpha](Q) = (463, 495)$. Also, $\mathbf{e}_5(Q, [\alpha]Q) = 89 \neq 1$. Again, this shows that $[\alpha]$ works as a distortion map.

Given these calculations it is not hard to find the matrix of the action of $[\alpha]$ on $E[5]$ relative to the basis P, Q

$$[\alpha] = \begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial of this matrix is irreducible modulo 5 and thus the action on $E[5]$ is irreducible.

Example 3.3. One can use the elliptic curve E from Example 3.2 to illustrate case (2b) of Theorem 2.1. This time we look at $E[2]$ (also contained in $\mathbb{F}_{701}$) which is generated by the points $P = (319, 0)$ and $Q = (389, 0)$. The prime 2 splits completely in $\mathrm{End}(E)$. The proof of Theorem 2.1 tells us that the characteristic polynomial of the action of the endomorphism $[\alpha]$ has two distinct roots and would work as a distortion map for all but two subgroups of $E[2]$. Now the minimal polynomial α is $x^2 - x + 2$ and modulo 2 this splits as $x(x + 1)$. Thus the action of $[\alpha]$ on $E[2]$ will have two eigenvectors, with eigenvalues 0 and 1 respectively. It is easy to check given the formula for $[\alpha]$ that indeed $[\alpha](P) = 0_E$ and $[\alpha](Q) = Q$.

Example 3.4. In this example we illustrate that case (1) of Theorem 2.1 also occurs. Consider the curve $E/\mathbb{Q}$ given by the Weierstrass equation

$$y^2 = x^3 - \frac{3375}{121}x + \frac{6750}{121}.$$

The j-invariant of E is $2^4 3^3 5^3$ and the conductor of E is 108900. E has CM by the order of conductor 2 in $\mathbb{Q}(\sqrt{-3})$. Thus $\mathrm{End}(E) \cong \mathbb{Z} + 2\mathcal{O}_K$ where $\mathcal{O}_K = \mathbb{Z} + \frac{1}{2}(1 + \sqrt{-3})\mathbb{Z}$. E has good reduction at the prime 13 and one sees that the reduction $\tilde{E}$ has $\mathbb{F}_{13}$-rational 2-torsion. Now $\mathrm{End}(\tilde{E}) \cong \mathrm{End}(E)$ by the Deuring reduction theorem ([Lan87] Chapter 13 §4, Theorem 12), but $\mathrm{End}(\tilde{E}) \mod 2 \cong (\mathbb{Z}/2\mathbb{Z})$ and so there are no distortion maps.

References

[BK98] Balasubramanian, R.; Koblitz, N.; *The improbability that an elliptic curve has subexponential discrete log problem under the Menezes-Okamoto-Vanstone Algorithm*, Journal of Cryptology, **Vol. 11**, No. 2, 141-145, 1998.

[GR04] Galbraith, S.; Rotger. V.; *Easy decision Diffie-Hellman groups*, LMS J. Comput. Math., **7**, 201-218, 2004.

[Lan87] Lang, S.; *Elliptic Functions*, 2nd ed., Graduate Texts in Math., vol. 112, Springer-Verlag, 1987.

[Sil94] Silverman, J.; *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Math. Vol. 151, Springer-Verlag, 1994.

[Ver01] Verheul, E., R.; *Evidence that XTR is more secure than supersingular elliptic curve cryptosystems*, EUROCRYPT 2001, Lecture Notes in Computer Science, **2045**, Springer-Verlag, 195-201, 2001.

[Ver04] Verheul, E., R.; *Evidence that XTR is more secure than supersingular elliptic curve cryptosystems (Journal version)*, Journal of Cryptology, **Vol. 17**, No. 4, 277-296, 2004.